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ANALYSIS OF SHALLOW CYLINDRICAL SHELL BY BOUNDARY ELEMENT METHOD

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Abstract—The objective of this article is to develop boundary element techniques for static analysis of cylindrical shell problems. The boundary integral equations are formulated by using the Maxwell-Betti reciprocal theorem. On the basis of the singular solutions for shallow cylindrical shells in complex variables proposed by Sanders and Simmonds, the displacements and tractions of the shell under various concentrated loadings are derived as the kernels in the boundary integrals near the singular points.

1. HISTORICAL REVIEW OF THE BOUNDARY ELEMENT METHODS FOR SHELLS

The boundary element technique, developed rapidly in recent years, has been recognized as a powerful approximate method in diverse boundary value problems. Its accuracy and efficiency build up its potential in becoming a significant solution method in engineering analysis. Efforts have been made to incorporate the boundary integral equation technique with modern computational methods in some thin shell problems (see for example, Tosaka and Miyake (1983), Forbes (1969), Fu and Harb (1990, 1991), Lu (1987), and Antes (1980)). Nevertheless, a comprehensive study of the boundary element method in the thin shell field is still far from achieved. The reason is due to the nature of how this method is formulated.

Basically, this solution method comprises two steps. The first is to formulate the boundary integral equations for a specific problem. The second is to seek the numerical solution to the established boundary integral equations. The advent of computers makes the second step an easier task. In fact, the term of boundary element stems from essentially the same discretion procedures as those used in the popular finite element method. Nevertheless, the major advantage of the boundary element method is that the dimension of the discretion is reduced by one, which significantly reduces the programming and computational efforts. Therefore, the formulation of the boundary integral equations for some important engineering problems demands further research.

The formulation of boundary integral equation for a problem depends on two factors. One is that an integral reciprocal relationship between two solutions to the governing differential equation should exist (see Cruse (1984)). The other is that there exists a known fundamental solution, i.e., a solution to the governing differential equation with a Dirac delta function being the 'force function'. In linear elasticity, including classical thin shell theory, the Maxwell-Betti reciprocal theorem provides the integral reciprocal relations, but the fundamental solution is problem dependent. To identify the fundamental solution is the key to success for applying the boundary element method on problems such as thin shells.

The effort of finding solutions of shells under concentrated load has been extended for a long period of time. Except for a few special cases, most of the solutions are for shallow shells. Among them, the solution to shallow spherical shells under normal point loads has the simplest form and has been employed as the kernel of boundary integral equation. However, the formulation of this method for shallow cylindrical shell has not been found. Most likely, the reason is the complexity of the fundamental solution for shallow cylindrical shells. The solution given by Jahanshahi (1963) only concerns normal load and is expressed in terms of the integral of the products of some special functions. Flugge and Elling's solutions (1972) concern normal and thermal loading and involve infinite series. Sanders and Simmonds (1970) gave solutions to both normal and membrane loading, expressed in modified Bessel functions and a function of integral form in complex variables. Despite the complexity of the solution form, the latter seems to be applicable in the formulation of boundary integral equations for a shallow cylindrical shell. Brooks (1988, 1991) employed these solutions in a boundary integral analysis of lugs attached to cylindrical pressure vessels.

Chernyshev (1963) showed that singularities of the solutions to a general shell under concentrated loads have the same nature as those of plates. However, as far as the formulation of the boundary integral equation is concerned, one needs only to know the nature of the singularity, that is, the behavior of the singular function when its arguments tend to the singular point. Chernyshev's results offered us the justification to formulate the boundary integral equations for a complete shell using the limit behavior of the singular solution of plate problems.

The implementation of the boundary integral equation method to a shell problem is another story. We can not use singular solutions to plate problem as the kernels of boundary integrals in a shell problem without penalty. The behaviors of the solution to shells differ from those of plates elsewhere except in the vicinity of a singular point. Hence, the displacements and tractions of the shallow cylindrical shell under various concentrated loadings are derived on the basis of the singular solutions in complex variables proposed by Sanders and Simmonds (1970). A boundary element scheme can be devised by using them as the kernels of the boundary integrals.

2. FORMULATION OF BOUNDARY INTEGRAL EQUATIONS FOR A SHELL PROBLEM

The boundary integral equations for a shell may be formulated based on Maxwell-Betti reciprocal theorem. Suppose that the solutions to a given cylindrical shell problem are characterized by F_i, V_i and corresponding boundary quantities ψ_n, M_n, Q_n, T_1 and T_2, where n denotes normal to the boundary. And suppose that an auxiliary equilibrium state of the same shell with the same boundary but different load and/or boundary quantities is given by F_i^*, V_i^*, ψ_n^*, M_n^*, T_1^*, and T_2^*. The Maxwell-Betti theorem states:

$$\int_A \sum_{j=1}^{3} F_j^* V_j \,\mathrm{d}A + \oint_\Gamma (M_b^* \Psi + Q^* w + T_1^* \mu + T_2^* v)\,\mathrm{d}s + \sum_l [M_v^*]_l w_l$$

$$= \int_A \sum_{i=1}^{3} F_j V_j^* \,\mathrm{d}A + \oint_\Gamma (M_b \Psi^* + Q w^* + T_1 u^* + T_2 v^*)\,\mathrm{d}s + \sum_l [M_v]_l w_l^* \quad (1)$$

where l ranges from 1 to the number of corners, $[M_v]_l$ means the jump of the twist moment in the tangential plane over a corner l, which is equivalent to the reaction force given by the support. T_1 and T_2 are membrane forces in the x and y directions, respectively.

Equation (1) gives the integral reciprocal relation between the solution to the problem and an auxiliary equilibrium state. In order to derive the boundary integral representation for the displacement field, it is necessary to choose the auxiliary state so that it represents the fundamental solution to the differential operators of the shell. In other words, we need to chose the auxiliary state as the solutions correspond to point loads acting on the shell in the direction of the desired displacement. It is obviously desirable to have the fundamental solution in close form. Unfortunately, such a solution has not been found due to the coupled nature and high order of the differential equations. However, knowing only the behavior of the fundamental solutions in the vicinity of the source point is enough for the

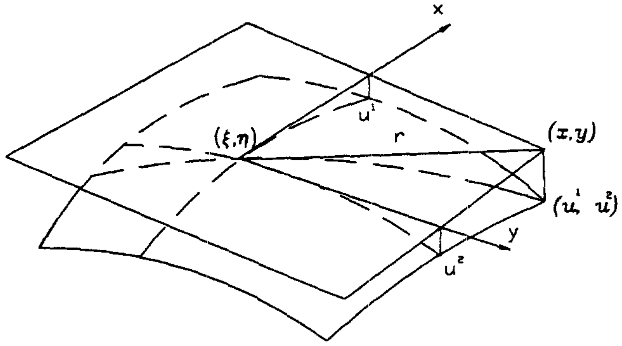

Fig. 1. A plate tangent to a shell.

formulation of the integral equation. The behavior of the fundamental solutions in the entire region is needed only for the calculation of the integrals. It has been proved by Chernyshev (1963) that the dominant part of the displacement and stress distribution in the neighborhood of the source point is the same as for the corresponding problem of a flat plate. Flügge (1972) pointed out that the stress distribution in the vicinity around the source point may be represented by a shallow shell that osculates the given shell at the source. The solutions to the shallow cylindrical shell obtained by Jahanshahi (1963) and Sanders (1973) again showed that the singularity of the solution to shells under the point loads in the vicinity of the source has the same nature as the singular solution to the corresponding plate problem. Thus, we are justified to use plate singularity in the region infinitely close to the source to obtain the boundary integral equation for the shell problem.

Suppose that there is a concentrated load acting at point (ξ, η) of the middle surface of the shell. A plate is tangent to the shell at (ξ, η). The geometric relation of the shell and the plate is given in Fig. 1. The displacements and stress resultants of this plate under the point load are given in Table 1.

In Table 1:

$$\mu = \frac{4\pi E h}{(1+\nu)(3-\nu)}, \quad b = \frac{2(1+\nu)}{1-\nu}, \quad \lambda = \frac{1-\nu}{4\pi}, \quad \gamma = \frac{1+\nu}{3-\nu}$$

$$r = \sqrt{(x-\xi)^2+(y-\eta)^2}, \quad r_x = \frac{\partial r}{\partial x}, \quad r_y = \frac{\partial r}{\partial y}, \quad r_n = \frac{\partial r}{\partial n}, \quad r_t = \frac{\partial r}{\partial t}$$

Table 1. Singular solution of plate extension and bending

	Point loading direction		
	x	y	z
u	$\frac{1}{\mu}(\ln r - \gamma r_x^2)$	$-\frac{1}{\mu}\gamma r_x r_y$	0
v	$-\frac{1}{\mu}\gamma r_x r_y$	$\frac{1}{\mu}(\ln r - \gamma r_y^2)$	0
w	0	0	$\frac{1}{8\pi D} r^2 \ln r$
T_1	$\frac{\lambda}{r}(1+br_x^2)r_n$	$\frac{\lambda}{r}(-r_t + br_x r_y r_n)$	0
T_2	$\frac{\lambda}{r}(r_t + br_x r_y r_n)$	$\frac{\lambda}{r}(1+br_y^2)r_n$	0
Q_n	0	0	$\frac{1}{4\pi}[2r_n + (1-\nu)(r_n - \kappa r)(r_n^2 - r_t^2)]$
M_n	0	0	$\frac{1}{4\pi}[(1+\nu)(1+2\ln r) + 2(r_n^2 + \nu r_t^2)]$

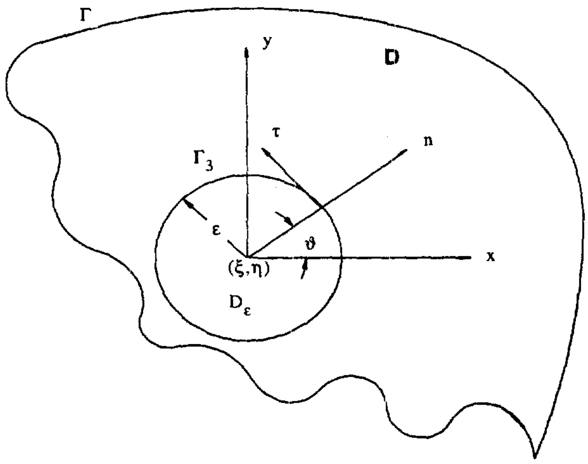

Fig. 2. A source at an interior point.

and κ is the curvature of a given curve at the point of interest. n and t denote tangent and normal of a given curve at the point of interest. Let F_j^*in the auxiliary state be a concentrated load at (ξ, η), which corresponds to a generalized delta function, and all loads in other directions be zero. The reciprocal eqn (1) is applicable in the domain D except at (ξ, η), D_ε, eqn (1) will give

$$\int_{D-D_\varepsilon} \sum_{j=1}^{3} F_{ij}^* V_j \, dA + \oint_{\Gamma+\Gamma_\varepsilon} (M_b^* \Psi_n + Q^* w + T_1^* u + T_2^* v) \, ds + \sum_l [M_v^*]_l w_l$$
$$= \int_{D-D_\varepsilon} \sum_{j=1}^{3} F_{ij} V_j^* \, dA + \oint_{\Gamma+\Gamma_\varepsilon} (M_b \Psi^* + Q w^* + T_1 u^* + T_2 v^*) \, ds + \sum_l [M_v]_l w_l^*. \quad (2)$$

The surface and line integrals in eqn (2) have certain characteristics similar to those of integral expressions of two dimensional potentials. Quantities relating to the auxiliary state serve as "kernels" of the integral expression. These kernels have some singularities at the source point (ξ, η). Let us first determine the limits of the integrals at an interior point (ξ, η) as shown in Fig. 2. Since the auxiliary load is zero everywhere in the region *D-De*, the first integral is zero on the left side of eqn (2). For computing the remaining integrals, rewrite

$$t_k = \begin{bmatrix} T_1 \\ T_2 \\ Q \\ M_b \end{bmatrix} \quad V_k = \begin{bmatrix} u \\ v \\ w \\ \Psi \end{bmatrix} \quad t_{ik} \begin{bmatrix} T_{1i}^* \\ T_{2i}^* \\ Q_i^* \\ M_i^* \end{bmatrix} \quad V_{ik}^* \begin{bmatrix} u_i^* \\ v_i^* \\ w_i^* \\ \Psi_i^* \end{bmatrix} \quad (3)$$

where the subscript i denotes the direction that the point load is acting along. On the circle with the center at the point of action of the concentrated load, one has

$$r = \varepsilon, \quad ds = \varepsilon \, d\theta, \quad r_x = \cos\theta, \quad r_y = \sin\theta, \quad r_n = 1, \quad r_t = 0, \quad \kappa = \varepsilon.$$

It is easy to see, upon using the value given in Table 1, that

$$\lim_{\varepsilon\to 0} \int_{D_\varepsilon} V_{ik}^* \, dA = \lim_{\varepsilon\to 0} \int_0^{2\pi} \int_0^{\varepsilon} V_{ik}^*(r, \theta) r \, dr \, d\theta = 0 \quad (4a)$$

$$\lim_{\varepsilon\to 0} \oint_{\Gamma_\varepsilon} V_{ik}^* \, ds = \lim_{\varepsilon\to 0} \int_0^{2\pi} V_{ik}^*(\varepsilon, \theta) \varepsilon \, d\theta = 0 \quad (4b)$$

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{11}^{*}\,\mathrm{d}s = \lambda\int_0^{2\pi}(1+b\cos^2\theta)\,\mathrm{d}\theta = 1 \tag{4c}$$

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{12}^{*}\,\mathrm{d}s = \lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{21}^{*}\,\mathrm{d}s = \lambda\int_0^{2\pi}\frac{b}{2}\sin 2\theta\,\mathrm{d}\theta = 0 \tag{4d}$$

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{22}^{*}\,\mathrm{d}s = \lambda\int_0^{2\pi}(1+b\sin^2\theta)\,\mathrm{d}\theta = 1 \tag{4e}$$

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{13}^{*}\,\mathrm{d}s = \lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{31}^{*}\,\mathrm{d}s = \lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{23}^{*}\,\mathrm{d}s = \lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{32}^{*}\,\mathrm{d}s = 0 \tag{4f}$$

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{33}^{*}\,\mathrm{d}s = \frac{1}{4\pi}\int_0^{2\pi}2\,\mathrm{d}\theta = 1. \tag{4g}$$

To compute $\oint_{\Gamma_\varepsilon} t_{34}^{*}\mathrm{d}s$ we project the moment M_n in two arbitrary directions, say x and y directions, then the resultant moment about x axis is

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} M_{3n}^{*}\cos\theta\,\mathrm{d}s = \lim_{\varepsilon\to 0}(\varepsilon\ln\varepsilon)\int_0^{2\pi}2\cos\theta\,\mathrm{d}\theta = 0.$$

Similarly, the resultant moment about the y axis is

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} M_{3n}^{*}\sin\theta\,\mathrm{d}s = \lim_{\varepsilon\to 0}(\varepsilon\ln\varepsilon)\int_0^{2\pi}2\sin\theta\,\mathrm{d}\theta = 0.$$

So, we have

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{34}^{*}\,\mathrm{d}s = 0. \tag{4h}$$

It is assumed that the functions F_k, t_k, and V_k related to the primary state are continuous up to second order in the domain and on the boundary. They can be expanded in Taylor series about the source $O(\xi,\eta)$

$$f = f_o + \frac{\partial f}{\partial u^1}\delta u^1 + \frac{\partial f}{\partial u^2}\delta u^2 + \mathrm{L}$$

where f may stand for F, V or t, and the subscripts for F, V and t have been omitted for convenience. By using this series expansion

$$\int_{D_\varepsilon} F_k V_{ik}^{*}\,\mathrm{d}A = \lim_{\varepsilon\to 0}\int_{D_\varepsilon}(F_o+\cdots)_k V_{ik}^{*}\,\mathrm{d}A = F_k(\xi,\eta)\lim_{\varepsilon\to 0}\int_{D_\varepsilon} V_{ik}^{*}\,\mathrm{d}A.$$

Similarly,

$$\oint_{\Gamma_\varepsilon} t_k V_{ik}^* \, dA = \lim_{\varepsilon\to 0} \oint_{\Gamma_\varepsilon} (t_o + \cdots)_k V_{ik}^* \, dA = t_k(\xi, \eta) \lim_{\varepsilon\to 0} \oint_{\Gamma_\varepsilon} V_{ik}^* \, dA$$

$$\oint_{\Gamma_\varepsilon} V_k t_{ik}^* \, dA = \lim_{\varepsilon\to 0} \oint_{\Gamma_\varepsilon} (V_o + \cdots)_k t_{ik}^* \, dA = V_i(\xi, \eta) \lim_{\varepsilon\to 0} \oint_{\Gamma_\varepsilon} t_{ik}^* \, dA.$$

Upon using eqn (4), eqn (2) leads to

$$V_i(\xi, \eta) = \int_A \sum_{j=1}^{3} F_j V_{ij}^* dA + \oint_\Gamma (M_b \Psi_i^* + Q w_i^* + T_1 u_i^* + T_2 v_i^*) \, ds$$
$$- \oint_\Gamma (M_{bi}^* \Psi + Q_i^* w_i + T_1^* u_i + T_2^* v_i) \, ds - \sum_l [M_v^*]_l w_l + \sum_l [M_v]_l w_{il}^*. \quad (5)$$

Or, by using the vector form given in (3) and summation notation

$$V_i(\xi, \eta) = \int_A F_j V_{ij}^* dA + \oint_\Gamma t_k V_{ik}^* \, ds - \oint_\Gamma t_{ik}^* V_k \, ds - [M_v^*]_l w_l + [M_v]_l w_l^* \quad (6)$$

where k ranges from 1 to 4, i and j range from 1 to 3.

Note that (ξ, η) is a point located inside the domain. When (ξ, η) is located at the boundary, the above procedure, that led to eqn (5), still applies except that the integration limits of θ are changed. For those points of action on smooth boundary, referring to Fig. 3, one has

$$\lim_{\varepsilon\to 0} \oint_{\Gamma_\varepsilon} t_{11}^* \, ds = \lambda \int_0^\pi (1 + b\cos^2\theta) \, d\theta = \frac{1}{2}$$

$$\lim_{\varepsilon\to 0} \oint_{\Gamma_\varepsilon} t_{22}^* \, ds = \lambda \int_0^\pi (1 + b\sin^2\theta) \, d\theta = \frac{1}{2}$$

$$\lim_{\varepsilon\to 0} \oint_{\Gamma_\varepsilon} t_{33}^* \, ds = \frac{1}{4\pi} \int_0^\pi 2 \, d\theta = \frac{1}{2}.$$

For those points of action on corners of the boundary, referring to Fig. 4, one has

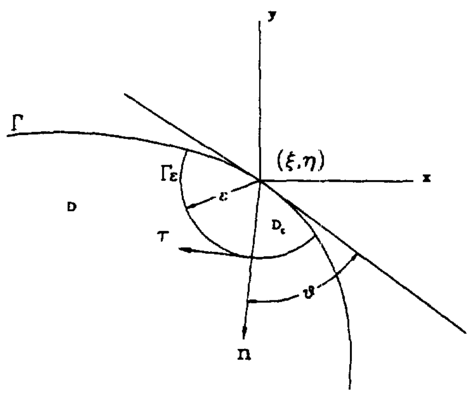

Fig. 3. A source at a smooth boundary point.

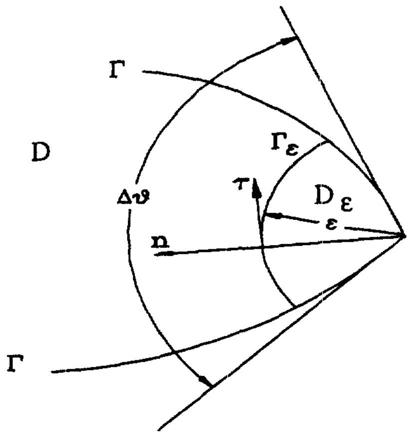

Fig. 4. A source at a boundary corner.

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{11}^* \, ds = \lambda \int_0^{\Delta\theta} (1+b\cos^2\theta)\, d\theta = \frac{\Delta\theta}{2\pi}$$

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{22}^* \, ds = \lambda \int_0^{\Delta\theta} (1+b\sin^2\theta)\, d\theta = \frac{\Delta\theta}{2\pi}$$

$$\lim_{\varepsilon\to 0}\oint_{\Gamma_\varepsilon} t_{33}^* \, ds = \frac{1}{4\pi}\int_0^{\Delta\theta} 2\, d\theta = \frac{\Delta\theta}{2\pi}$$

where $\Delta\theta$ is the angle spanned by the two adjacent sides of the corner inside the region. The remaining integrals in eqn (4) for boundary points are all zeros. Thus for boundary points, eqn (2) leads to

$$CV_i(\xi,\eta) = \int_A F_j V_{ij}^* \, dA + \oint_\Gamma t_k V_{ik}^* \, ds - \oint_\Gamma t_{ik}^* V_k \, ds - [M_v^*]_l w_l + [M_v]_l w_l^* \tag{7}$$

where C is a constant given as follows:

$$C = \begin{cases} 1/2 & \text{when}(\xi,\eta) \text{ is on a smooth boundary} \\ \Delta\theta/2\pi & \text{when}(\xi,\eta) \text{ is on a boundary corner} \end{cases}$$

The above integral eqns (7) and (6) have great significance. First, when the boundary quantities of a given problem are known, eqn (6) supplies all displacements inside the region through the boundary integration, as well as all desired stress resultant that can be obtained by differentiating the displacements expressed by eqn (5), because the displacements are continuous with respect to collocation variables (ξ, η). Second, eqn (7) offers a set of boundary integral equations that can be numerically solved for all necessary boundary quantities themselves. However, there are four unknown boundary quantities at a regular boundary point and only three equations are given in eqn (7). So one more equation is needed. It can be obtained either by finding the solutions of an auxiliary state that is under a concentrated couple acting in desired directions or by deriving the rotation of the normal about boundary tangent, Ψ, from eqn (6). The latter is employed here.

With $j = 2$ and 3, eqn (6) gives the normal and circumferential displacements at a given point (ξ, η),

$$w(\xi,\eta) = \int_A F_j V^*_{3j}\,\mathrm{d}A + \oint_\Gamma t_k V^*_{3k}\,\mathrm{d}s - \oint_\Gamma t^*_{3k} V_k\,\mathrm{d}s - [M^*_{v3}]_l w_l + [M_v]_l w^*_{3l}$$

$$v(\xi,\eta) = \int_A F_j V^*_{2j}\,\mathrm{d}A + \oint_\Gamma t_k V^*_{2k}\,\mathrm{d}s - \oint_\Gamma t^*_{2k} V_k\,\mathrm{d}s - [M^*_{v2}]_l w_l + [M_v]_l w^*_{2l}.$$

The rotation vector can be written as:

$$\Psi = n^\alpha q_\alpha = n^\alpha (d^\beta_\alpha v_\beta + w_\alpha) = n_1 \frac{\partial w}{\partial \xi} + n_2 \left(\frac{\partial w}{\partial \eta} - \frac{v}{R}\right).$$

The derivative with respect to a given direction, say $(\partial w)/(\partial \xi)$, is

$$\frac{\partial w(\xi,\eta)}{\partial \xi} = \int_A F_j \frac{\partial V^*_{3j}}{\partial \xi}\,\mathrm{d}A + \oint_\Gamma t_k \frac{\partial V^*_{3k}}{\partial \xi}\,\mathrm{d}s - \oint_\Gamma V_k \frac{\partial t^*_{3k}}{\partial \xi}\,\mathrm{d}s - \left[\frac{\partial M^*_{v3}}{\partial \xi}\right]_l w_l + [M_v]_l \frac{\partial w^*_{3l}}{\partial \xi}$$

since only the quantities associated to the auxiliary state are functions of (ξ,η). Upon moving the point (x, h) to boundary and letting the direction v be the outward normal to the boundary, we have

$$C\frac{\partial w(\xi,\eta)}{\partial \xi} = \int_A F_j \frac{\partial V^*_{3j}}{\partial \xi}\,\mathrm{d}A + \oint_\Gamma t_k \frac{\partial V^*_{3k}}{\partial \xi}\,\mathrm{d}s - \oint_\Gamma V_k \frac{\partial t^*_{3k}}{\partial \xi}\,\mathrm{d}s - \left[\frac{\partial M^*_{v3}}{\partial \xi}\right]_l w_l + [M_v]_l \frac{\partial w^*_{3l}}{\partial \xi}. \quad (8)$$

By denoting

$$V^*_{4k} = n_1 \frac{\partial V^*_{3k}}{\partial \xi} + n_2 \left(\frac{\partial V^*_{3k}}{\partial \eta} - \frac{V^*_{2k}}{R}\right), \quad t^*_{4k} = n_1 \frac{\partial t^*_{3k}}{\partial \xi} + n_2 \left(\frac{\partial t^*_{3k}}{\partial \eta} - \frac{t^*_{2k}}{R}\right) \quad (9)$$

we will have

$$C\Psi = \int_A F_j V^*_{4j}\,\mathrm{d}A + \oint_\Gamma t_k V^*_{4k}\,\mathrm{d}s - \oint_\Gamma V_k t^*_{4k}\,\mathrm{d}s - \left[n_1 \frac{\partial M^*_{v3}}{\partial \xi} + n_2 \left(\frac{\partial M^*_{v3}}{\partial \eta} - \frac{M^*_{v2}}{R}\right)\right]_l w_l + [M_v]_l \left[n_1 \frac{\partial w^*_3}{\partial \xi} + n_2 \left(\frac{\partial w^*_3}{\partial \eta} - \frac{v^*_2}{R}\right)\right]. \quad (10)$$

Equations (6), (7), and (10) supply enough information to solve a given boundary problem of shells.

3. SOLUTIONS TO SHALLOW CYLINDRICAL SHELL UNDER CONCENTRATED LOADS

The basic equations and the relations of the quantities to describe a shallow shell are listed here for consistency. The shell is said to be shallow when the conditions

$$(\partial z/\partial x)^2 \ll 1$$

$$(\partial z/\partial y)^2 \ll 1$$

are satisfied everywhere on the middle surface. In the basic equations, the squares and products of the derivatives $\partial z/\partial x$ and $\partial z/\partial y$ may be neglected in comparison with unity.

When analyzing shallow shells, it is convenient to employ the parametric representation that has the same form of the Cartesian description for the middle surface. That is

$$x^1 = x, \quad x^2 = y, \quad x^3 = z(x,y). \tag{11}$$

After simplification, the metric tensor, $a_{\alpha\beta}$, and curvature tensor, $d_{\alpha\beta}$, become

$$a_{\alpha\beta} = \delta_\alpha^\beta, \quad d_{\alpha\beta} = \frac{\partial^2 z}{\partial x \partial y} = z_{,\alpha\beta}.$$

Strain-displacement relations

$$E_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}) - z_{,\alpha\beta} w \tag{12}$$

$$K_{\alpha\beta} = w_{,\alpha\beta}. \tag{13}$$

The rotation vector will be $a_\alpha = w_{,\alpha}$.

The equilibrium equations

The application of the simplified expressions for the bending measures and rotations implies that the equations of equilibrium should be simplified correspondingly. It can be shown that the principle of virtual work is equivalent to a set of approximate equilibrium equations together with approximate expressions for the boundary tractions in terms of effective stress resultants.

$$N_{\alpha\beta,\beta} + F_\alpha = 0 \tag{14a}$$

$$M_{\alpha\beta,\alpha\beta} - z_{,\alpha\beta} N_{\alpha\beta} - p = 0. \tag{14b}$$

The boundary traction expressions

The boundary tractions of a shallow shell can be simplified as

$$\begin{aligned} M_b &= M_{\alpha\beta} n_\alpha n_\beta \\ Q &= -M_{\alpha\beta,\alpha} n_\beta - \frac{\partial M_{\alpha\beta}}{\partial s} n_\alpha t_\beta \\ \hat{T}_\alpha &= N_{\alpha\beta} n_\beta. \end{aligned} \tag{15}$$

The constitutive relations

The constitutive relations for a shallow shell are

$$E_{\alpha\beta} = \frac{1}{Eh}[(1+\nu)N_{\alpha\beta} - \nu\delta_\alpha^\beta N_{\gamma\gamma}] \tag{16a}$$

$$M_{\alpha\beta} = D[(1-\nu)K_{\alpha\beta} + \nu\delta_\alpha^\beta K_{\gamma\gamma}]. \tag{16b}$$

The compatibility equations

$$\varepsilon_{\alpha\gamma}\varepsilon_{\rho\beta} K_{\alpha\beta,\gamma} = 0 \tag{17a}$$

$$\varepsilon_{\alpha\gamma}\varepsilon_{\rho\beta}(z_{,\rho\gamma} K_{\alpha\beta} + E_{\alpha\beta,\rho\gamma}) = 0. \tag{17b}$$

By defining

$$\bar{K}_{\gamma\rho} = \varepsilon_{\alpha\gamma}\varepsilon_{\rho\beta}K_{\alpha\beta}$$
$$\bar{E}_{\gamma\rho} = \varepsilon_{\alpha\gamma}\varepsilon_{\rho\beta}E_{\alpha\beta} \tag{18}$$

we may write the compatibility eqns (17) in the following form

$$\bar{K}_{\gamma\rho,\gamma} = 0 \tag{19a}$$
$$\bar{E}_{\gamma\rho,\gamma\rho} + z_{,\gamma\rho}\bar{K}_{\gamma\rho} = 0. \tag{19b}$$

The shallow shell equations in complex variables

The resemblance between the compatibility eqns (19) and the homogeneous version of equilibrium eqns (14) hints the existence of the stress functions. Analogous to strain-displacement relations (12) and (13), one may introduce the following stress functions

$$\bar{N}_{\alpha\beta} = \Phi_{,\alpha\beta}$$
$$\bar{M}_{\alpha\beta} = \frac{1}{2}(\chi_{\alpha,\beta} + \chi_{\beta\alpha}) - z_{,\alpha\beta}\Phi \tag{20}$$

where

$$\bar{N}_{\alpha\beta} = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}N_{\gamma\delta}$$
$$\bar{M}_{\alpha\beta} = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}M_{\gamma\delta}. \tag{21}$$

All the variables are made dimensionless in following way (the dimensionless form precedes the colon):

$$x_\alpha : Lx_\alpha \quad u_\alpha : \frac{\sigma L}{E}u_\alpha \quad \phi : \sigma hL^2\phi z : \frac{L^2}{R}z \quad w : \frac{4\lambda^2\sigma R}{E}w \quad \chi_\alpha : \frac{\sigma hL^2}{4\lambda^2 R}\chi_\alpha$$

$$N_{\alpha\beta} : \sigma hN_{\alpha\beta} \quad E_{\alpha\beta} : \frac{\sigma}{E}E_{\alpha\beta} \quad p_\alpha : \frac{\sigma h}{L}p_\alpha M_{\alpha\beta} : \frac{\sigma hL^2}{4\lambda^2 R}M_{\alpha\beta} \quad K_{\alpha\beta} : \frac{\sigma R}{EL^2}K_{\alpha\beta} \quad p : \frac{\sigma h}{4\lambda^2 R}p.$$

Here L is a reference length, R is a reference radius of curvature, σ is a reference stress, E is Young's modulus, h is the constant shell thickness, and λ is a parameter defined by

$$\lambda^2 = \frac{L^2}{2Rh}\sqrt{3(1-\nu^2)}.$$

For the purpose of abbreviation, all the dependent variables in the above basic equations of shallow shells are considered as complex valued functions of x and y, that is

$$w = w' + iw''$$
$$N_{\alpha\beta} = N'_{\alpha\beta} + iN''_{\alpha\beta}$$

etc. Again, based on the similarity in form between the stress-equilibrium and the strain-compatibility equations, the following relations are attached to reduce the number of equations and the order of the governing differential equations

$$\Phi = -iw, \quad \chi_\alpha = -iu_\alpha - 2\nu w_{,\alpha}$$

which, with eqns (20) and (21), entail the following additional relations

$$N_{\alpha\beta} = i\bar{K}_{\alpha\beta} \tag{22}$$

$$M_{\alpha\beta} = -i\bar{E}_{\alpha b} + 2v\bar{K}_{\alpha\beta}. \tag{23}$$

In complex form the compatibility equations and the equilibrium equations give the following governing differential equations

$$\nabla^4 w + 4i\lambda^2 \bar{z}_{,\alpha\beta} w_{,\alpha\beta} = p + ivp_{\alpha,\alpha} \tag{24}$$

$$\frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) = i[(1+v)w_{,\alpha\beta} - \delta_{\alpha\beta}\Delta w - 4i\lambda^2 z_{,\alpha\beta} w] \tag{25}$$

where

$$\bar{z}_{,\alpha\beta} = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} z_{,\gamma\delta}.$$

The solutions of shallow shell equation under concentrated loading

When p_α and p are replaced by delta functions, a procedure of solutions to (24) and (25) were given by Sanders. In the case of shallow cylindrical shells the solutions are

1. Normal concentrated loading

$$w = \frac{i^{3/2}}{4\pi\lambda}(T + yR) \tag{26}$$

$$u = \frac{i}{2\pi}[U + y(S - \pi + \theta)] - \frac{i^{1/2}(1+v)}{4\pi\lambda}\sin\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r) \tag{27}$$

$$v = -\frac{i^{3/2}\lambda}{2\pi}y(T + yR) - \frac{i}{2\pi}x(S - \pi + \theta) + \frac{i}{2\pi}\cos\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r) - \frac{i^{1/2}(1+v)}{4\pi\lambda}R \tag{28}$$

where

$$R = \frac{1}{2}[P(\lambda r, \theta) + P(\lambda r, \pi - \theta)] \tag{29}$$

$$S = \frac{1}{2}[P(\lambda r, \theta) - P(\lambda r, \pi - \theta)] \tag{30}$$

$$T = x\sin\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r) + r\cos\text{h}(i^{1/2}\lambda x)K_1(i^{1/2}\lambda r) \tag{31}$$

$$U = x\cos\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r) + r\sin\text{h}(i^{1/2}\lambda x)K_1(i^{1/2}\lambda r) \tag{32}$$

$$P(r, \theta) = \sin\theta\int_0^\rho e^{s\cos\theta}K_0(s)\,\text{d}s \tag{33}$$

where r and θ are polar coordinate of x-y plane, $\rho = i^{1/2}r$, K_0 and K_1 are the Bessel functions of the second kind.

2. Tangential concentrated force in the x direction

$$w = -\frac{i}{2\pi}[U+y(S-\pi+\theta)]+\frac{i^{1/2}(1+\nu)}{4\pi\lambda}\sin\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r) \tag{34}$$

$$u = -\frac{i^{1/2}\lambda}{\pi}(T+yR)+\frac{(1+\nu)(3-\nu)}{4\pi}\cos\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r)$$
$$-\frac{(1+\nu)^2}{4\pi}\frac{x}{r}\sin\text{h}(i^{1/2}\lambda x)K_1(i^{1/2}\lambda r) \tag{35}$$

$$v = \frac{i\lambda^2}{\pi}y(U+y(S-\pi+\theta)+\frac{i^{1/2}\lambda}{\pi}[xR-y\sin\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r)$$
$$+\frac{(1+\nu)^2}{4\pi}\frac{y}{r}\sin\text{h}(i^{1/2}\lambda x)K_1(i^{1/2}\lambda r). \tag{36}$$

3. Tangential concentrated loading in y direction

$$w = \frac{i^{3/2}\lambda}{2\pi}y(T+yR)+\frac{i}{2\pi}x(S-\pi+\theta)-\frac{i}{2\pi}\cos\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r)+\frac{i^{1/2}(1+\nu)}{4\pi\lambda}R \tag{37}$$

$$u = \frac{i\lambda^2}{\pi}y(U+y(S-\pi+\theta)+\frac{i^{1/2}\lambda}{\pi}[xR-y\sin\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r)$$
$$+\frac{(1+\nu)^2}{4\pi}\frac{y}{r}\sin\text{h}(i^{1/2}\lambda x)K_1(i^{1/2}\lambda r) \tag{38}$$

$$v = -\frac{2i\lambda^2}{\pi}x(U+y(S-\pi+\theta)-\frac{i^{1/2}\lambda}{\pi}(1+2\nu+\frac{2}{3}i\lambda^2y^2)(T+yR)$$
$$+\frac{1}{4\pi}\left[(1+\nu)(3-\nu)+\frac{8}{3}i\lambda^2r^2\right]\cos\text{h}(i^{1/2}\lambda x)K_0(i^{1/2}\lambda r)$$
$$-\frac{2i^{1/2}\lambda}{3\pi}r\cos\text{h}(i^{1/2}\lambda x)K_1(i^{1/2}\lambda r)-\frac{1}{4\pi}\left[(1+\nu)^2-\frac{8}{3}i\lambda^2r^2\right]\frac{x}{r}\sin\text{h}(i^{1/2}\lambda x)K_1(i^{1/2}\lambda r) \tag{39}$$

4. The computation of the function P and its derivatives
The derivatives of the function P may be obtained from eqn (33):

$$\frac{\partial P}{\partial r} = i^{1/2}\sin\theta e^{\sqrt{ix}}K_0(\rho) \tag{40}$$

$$\frac{\partial P}{\partial \theta} = -1+i^{1/2}e^{\sqrt{ix}}[rK_1(\rho)+xK_0(\rho)] \tag{41}$$

$$\frac{\partial P}{\partial x} = \frac{y}{r^2}[1-i^{1/2}e^{\sqrt{ix}}K_1(\rho)] \tag{42}$$

$$\frac{\partial P}{\partial y} = i^{1/2}e^{\sqrt{ix}}K_0(\rho)-\frac{x}{r^2}[1-i^{1/2}e^{\sqrt{ix}}rK_1(\rho)]. \tag{43}$$

The most difficult part comes to the computation of function P itself. It has been shown by Sanders and Simmonds (1970) that

$$P \approx -\theta+\pi \operatorname{erf}\zeta-\sqrt{\pi}\zeta e^{-\zeta^2}\left[2i^{3/2}(8r)^{-1}+i(2\zeta^2-3)(8r)^{-2}\right.$$
$$\left.+i^{1/2}(4\zeta^4-8\zeta^2+15)(8r)^{-3}+\left(10\zeta^6-29\zeta^4+\frac{33}{2}\zeta^2-\frac{525}{4}\right)(8r)^{-4}+\cdots\right] \quad (44)$$

where

$$\zeta = i^{1/4}(r-x)^{1/2} = i^{1/4}(2r)^{1/2}\sin\frac{1}{2}\theta. \quad (45)$$

For very small r

$$P \approx i^{1/2} y \ln r. \quad (46)$$

For very large r

$$P \approx \pi-\theta \quad \left(x \leqslant 0, \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}\right)$$

$$P \approx -\theta+\pi \operatorname{erf}\left(\frac{i^{1/4}y}{\sqrt{2x}}\right)-\frac{1}{4}i^{1/4}\sqrt{\frac{\pi}{2x}}\frac{y}{x}\left(1-i^{1/2}\frac{y^2}{x}\right)e^{-\sqrt{i}y^2/2x}\left(x>0,\ -\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}\right). \quad (47)$$

4. BOUNDARY ELEMENT FORMULATION

It is nearly impossible to seek their theoretical solutions of eqns (6) and (7). However, they can be solved numerically. The boundary of the shell is divided into N segments or elements as shown in Fig. 5. A quadratic interpolation scheme is employed for all the boundary quantities, including the displacements, u, v, w and Ψ, and the boundary forces, $\hat{T}_1$, $\hat{T}_2$, Q, and M_b. That is, three nodes are set for each element, one at each end and one somewhere at the middle of the element. Let the nodal value of the boundary quantities be expressed in the vector form:

$$\mathbf{v}^i = [u^i v^i w^i \Psi^i]^T \quad \mathbf{t}^i = [\hat{T}_1^i \hat{T}_2^i Q^i M^i].$$

Then the displacements and boundary tractions over the element are interpolated quadratically as follows:

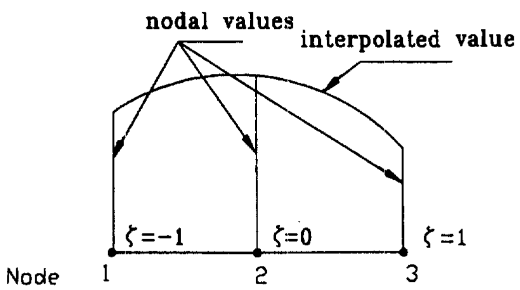

Fig. 5. The quadratic interpolation.

$$\mathbf{V} = \begin{Bmatrix} u \\ v \\ w \\ \Psi \end{Bmatrix} = \sum_{i=1}^{3} \mathbf{v}^i \Phi_i \quad \mathbf{t} = \begin{Bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ Q \\ M \end{Bmatrix} = \sum_{i=1}^{3} \mathbf{t}^i \phi_i$$

where

$$\phi_1 = \frac{1}{2}\zeta(\zeta-1), \quad \phi_2 = 1-\zeta^2, \quad \phi_3 = \frac{1}{2}\zeta(\zeta+1)$$

and ζ is the local parameter defining the boundary element geometry as shown in Fig. 5.

After the discretization eqns (6) and (7) can be written as

$$c^i\mathbf{v}^i = \sum_{j=1}^{N}\sum_{k=1}^{m}\left(\int_{\Gamma_j}\mathbf{v}^*\phi_k\,\mathrm{d}\Gamma\right)\mathbf{t}^k - \sum_{j=1}^{N}\sum_{k=1}^{m_j}\left(\int_{\Gamma_j}\mathbf{t}^*\phi_k\,\mathrm{d}\Gamma\right)\mathbf{v}^k - \sum_{l=1}^{nc}[\mathbf{M}_v^{*l}]w_l + \sum_{l=1}^{nc}[M_v^l]\mathbf{w}_l^* + \sum_{s=1}^{M}\int_{\Omega_s}\mathbf{u}^*\mathbf{F}\,\mathrm{d}\Omega \qquad (48)$$

where N is the number of nodes, m_j is the number of nodes that the j-th element has, nc is the number of corners. The indices in bold face are global number and those in italic face are local number attached to an element.

$$\mathbf{v}^* = \begin{bmatrix} V_{11}^* & V_{12}^* & V_{13}^* & V_{14}^* \\ V_{21}^* & V_{22}^* & V_{23}^* & V_{24}^* \\ V_{31}^* & V_{32}^* & V_{33}^* & V_{34}^* \\ V_{41}^* & V_{42}^* & V_{43}^* & V_{44}^* \end{bmatrix} \quad \mathbf{t}^* = \begin{bmatrix} t_{11}^* & t_{12}^* & t_{13}^* & t_{14}^* \\ t_{21}^* & t_{22}^* & t_{23}^* & t_{24}^* \\ t_{31}^* & t_{32}^* & t_{33}^* & t_{34}^* \\ t_{41}^* & t_{42}^* & t_{43}^* & t_{44}^* \end{bmatrix}$$

$$\mathbf{u}^* = \begin{bmatrix} V_{11}^* & V_{12}^* & V_{13}^* \\ V_{21}^* & V_{22}^* & V_{23}^* \\ V_{31}^* & V_{32}^* & V_{33}^* \\ V_{41}^* & V_{42}^* & V_{43}^* \end{bmatrix} \quad \mathbf{F} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \quad [\mathbf{M}_v^*] = \begin{Bmatrix} [M_{v1}^*] \\ [M_{v2}^*] \\ [M_{v3}^*] \\ [M_{v4}^*] \end{Bmatrix} \quad \mathbf{w}^* = \begin{Bmatrix} V_{13}^* \\ V_{23}^* \\ V_{33}^* \\ V_{43}^* \end{Bmatrix}$$

and

$$\mathbf{c}^i = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix}^i$$

are the coefficient matrices depending on the geometry of the boundary where the node is located. $[\mathbf{M}_v] = \mathbf{M}_v^+ - \mathbf{M}_v^-$ is the jump of the twist moments over a corner. The subscript k in $[\mathbf{M}_{vk}^*]$ denotes the direction of the unit point load that causes the twist moment, and

$$[M_{v4}^*] = \left[n_1\frac{\partial M_{v3}^*}{\partial\xi} + n_2\left(\frac{\partial M_{v3}^*}{\partial\eta} - \frac{M_{v2}^*}{R}\right)\right].$$

The foregoing eqn (48) may be rearranged using all global indices. Define

$$\hat{\mathbf{H}}^{ik} = \begin{bmatrix} \hat{H}^*_{11} & \hat{H}^*_{12} & \hat{H}^*_{13} \\ \hat{H}^*_{21} & \hat{H}^*_{22} & \hat{H}^*_{23} \\ \hat{H}^*_{31} & \hat{H}^*_{32} & \hat{H}^*_{33} \\ \hat{H}^*_{31} & \hat{H}^*_{42} & \hat{H}^*_{43} \end{bmatrix}^{ik} = \Sigma_e \int_{\Gamma_e} \mathbf{t}^* \phi^e_k \, d\Gamma,$$

$$\mathbf{G}^{ik} = \begin{bmatrix} G^*_{11} & G^*_{12} & G^*_{13} \\ G^*_{21} & G^*_{22} & G^*_{23} \\ G^*_{31} & G^*_{32} & G^*_{33} \\ G^*_{41} & G^*_{42} & G^*_{43} \end{bmatrix}^{ik} = \sum_e \int_{\Gamma_e} \mathbf{v}^* \phi^e_k \, d\Gamma$$

where subscript i represents the source point number, i.e., all the calculations are referring to this source point i, subscript k represents the node number, and Σ_e means a summation over all those elements who share the same node k. ϕ^e_k is the interpolating function associated with node k in the e-th element. Then eqn (48) may be rewritten as

$$\mathbf{c}^i\mathbf{v}^i + \sum_{k=1}^{N} \hat{\mathbf{H}}^{ik}\mathbf{v}^k = \sum_{k=1}^{Ng} \mathbf{G}^{ik}\mathbf{t}^k - \sum_{l=1}^{nc} [\mathbf{M}^{*l}_v]w_l + \sum_{l=1}^{nc} [M^l_v]\mathbf{w}^*_l + \sum_{s=1}^{M} \int_{\Omega_s} \mathbf{u}^*\mathbf{F} \, d\Omega. \tag{49}$$

On the smooth part of the boundary, there are eight boundary quantities, four displacements and four tractions, at each node. Four of them are given, that is, either u or $\hat{T}_1$, either v or $\hat{T}_2$, either w or Q, either Ψ or M_b are given, and others are unknown. Due to the discontinuities of the tractions and the existence of the corner force, there are thirteen boundary quantities at each corner, that is, in addition to four displacements there are four tractions on each side of the node and a corner force, which equal in quantity to the jump of the twist moment, M_v, over the corner. However, the unknowns will generally remain four as in most commonly seen boundary combinations. To reflect the traction discontinuities we setup two nodal traction sets at each corner, one in each side of the corner. So we have $Ng = N+nc$ sets of boundary nodal values for tractions, where nc is the number of corner points.

Now let

$$\mathbf{H}^{ij} = \begin{cases} \hat{\mathbf{H}}^{ij} & i \neq j \\ \hat{\mathbf{H}}^{ij} + \mathbf{c}^i & i = j \end{cases}.$$

The system of equations become

$$\sum_{k=1}^{N} \mathbf{H}^{ik}\mathbf{v}^k = \sum_{k=1}^{Ng} \mathbf{G}^{ik}\mathbf{t}^k - \sum_{l=1}^{nc} [M^{*l}_v]w_l + \sum_{l=1}^{nc} [M^l_v]\mathbf{w}^*_l + \sum_{s=1}^{M} \int_{\Omega_s} \mathbf{u}^*\mathbf{F} \, d\Omega.$$

After the boundary conditions are incorporated in the above equation, shift all items associated with unknown boundary quantities into the left side and all other given values bounded together into the right side. Then the system of equation has the form

$$\Sigma_j \mathbf{A}^{ij}\mathbf{x}^i = \mathbf{F}^i \tag{50}$$

where $\mathbf{x}^j$ is either $\mathbf{v}^i$ or $\mathbf{t}^j$, $\mathbf{A}^{ij}$ is the associated matrix $\mathbf{H}^{ij}$ or $\mathbf{G}^{ij}$ with $\mathbf{x}^j$, depending on which value is an unknown, and

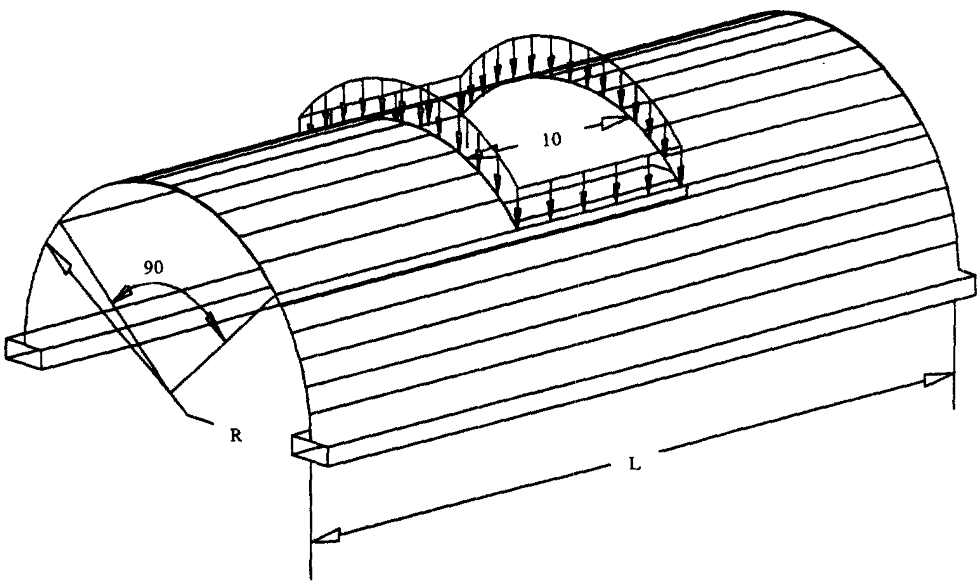

Fig. 6. A deep notched cylindrical shell.

$$\mathbf{F}^i = \sum_{s=1}^{M} \int_{\Omega_s} \mathbf{u}^* \mathbf{F} \, d\Omega + \sum_{\text{all given } t^p} \mathbf{G}^{ip} \mathbf{t}^p + \sum_{\text{all given } v^p} \mathbf{G}^{ip} \mathbf{v}^p.$$

After solving the system of eqn (50) for the unknown boundary quantities, we may obtain the displacements and stress resultants anywhere in the domain through eqns (6), (7) and (10) and their derivatives by using the same numerical scheme as described above.

5. EXAMPLE OF SHALLOW CYLINDRICAL SHELLS

An example of numerical solutions of shallow cylindrical shell problems obtained by the proposed method are presented here. The example selected is fairly simple in geometry. Even so, there is no theoretical solutions can be found for non-axisymmetrical loading. Thus, the results are compared with the finite element methods. The finite element solution is obtained using the shell element by SAP IV, (Bathe (1973)). Shown in Fig. 6 is a deep notched shell supported at two fixed edges and loaded by normal line loads along the notch. The dimensions are set: $L = 40$ feet, $R = 10$ feet, thickness of the shell $= 1.2$ inches, and the line load $= 1000$ lb/ft. For simplicity, the problem used a fictional isotropic homogeneous materials with modulus of elasticity being 4,176,000 psi and Poison ratio being 0.3.

To verify the results, this shell problem is also solved by SAP IV. The comparison is made in graphical form. The comparison of them with the finite element solutions is given in Fig. 7. The graph shows an overall agreement of the proposed method with the finite element method.

6. SUMMARY

The formulation of the boundary integral equation for the shell problem is described. All the required kernels of the integral equation are developed based on the singular solutions proposed by Sanders and Simmonds. Numerical procedures for solving the formulated boundary integral equations are established and an example problem is given. The results obtained for various shell flexure and extension problems are in overall agreement with the finite element solutions.

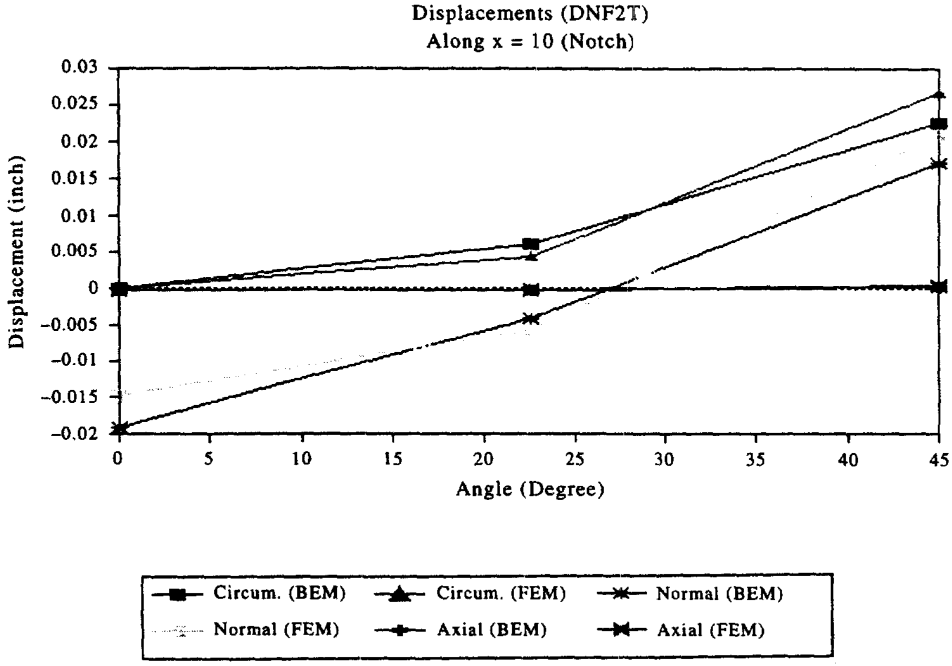

Fig. 7. Displacements along the circumferential edge of the notch.

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